BACKPAPER EXAMINATION
M. MATH I YEAR, II SEMESTER 2015-2016
COMPLEX ANALYSIS

Max. 100. Time limit: 3hrs

Notations: U is the open unit disc, $T = \partial U$, Log is the principal logarithm.

1. If f is an entire function such that for each complex number c there is a positive integer n with $f^{(n)}(c) = 0$. Show that f is a polynomial. [15]

2. Let $f(z) = \frac{1+iz}{1-iz}$ if $\operatorname{Im} z > 0$ and $f(z) = \frac{1-iz}{1+iz}$ if $\operatorname{Im} z < 0$. Show that f is holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and that we cannot extend it to a holomorphic function on any larger open set. [15]

3. Show that $Log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...$ for $|z| < 1$. [15]

Hint: there is a familiar power series related to above series.

4. Suppose f and g have poles at a point c. If $\lim\limits_{z \to c} \frac{f'(z)}{g'(z)}$ exists show that $\lim\limits_{z \to c} \frac{f(z)}{g(z)}$ exists and equals $\lim\limits_{z \to c} \frac{f'(z)}{g'(z)}$. [15]

5. Let $f(z) = \frac{z^2}{2} - \frac{z^3}{(2)(3)} + \frac{z^4}{(3)(4)} - ....$ Show that f is holomorphic in U but it can be extended to a holomorphic function on $\{z : \operatorname{Re} z > -1\}$. [15]

6. Prove that $\log|z|$ is harmonic in $\mathbb{C}\backslash\{0\}$. Does there exist a real valued harmonic function v on $\mathbb{C}\backslash\{0\}$ such that $\log|z| + iv(z)$ is analytic? If the answer is yes find one such function. Otherwise prove that there is no such function. [15]

7. Let $\phi \in L^1(\mathbb{R})$ and $u(x,y) = \int\limits_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\phi(t)dt$. Show that ϕ is well defined and harmonic on $\{z : \operatorname{Im} z > 0\}$. [10]

Hint: consider $\int\limits_{-\infty}^{\infty} \frac{1}{z-t}\phi(t)dt$

8. Let $\psi : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function and $u(x,y) = \psi(\sqrt{x^2+y^2}) = \psi(r)$ where, as usual, $r = \sqrt{x^2+y^2}$. Compute $\Delta u =$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in terms of derivatives of ψ with respect to r. Use your answer to find all twice continuously differentiable functions ϕ such that $\phi(\log|z|)$ is harmonic on $\mathbb{C}\backslash\{0\}$.

[15]